

# $Z_2$ -INDEX OF THE GRASSMANIAN $G_{2n}^n$

R.N. KARASEV

ABSTRACT. We study the real Grassmann manifold  $G_{2n}^n$  (of  $n$ -subspaces in  $\mathbb{R}^{2n}$ ), and the action of  $Z_2$  on it by taking the orthogonal complement. The homological index of this action is estimated from above and from below. In case  $n$  is a power of two it is shown that  $\text{ind } G_{2n}^n = 2n - 1$ .

## 1. INTRODUCTION

The topology of real Grassmannians has many applications in the discrete and convex geometry. For example, the Schubert calculus and other topological facts (e.g. from [3, 5]) can be applied to obtain some existence theorems for flat transversals (affine flats intersecting all members of a given family of sets), see [4, 15, 8, 12] for example.

In this paper we consider the Grassmannian  $G_{2n}^n$  of  $n$ -dimensional subspaces of  $\mathbb{R}^{2n}$ . This space has a natural  $Z_2$ -action (involution) by taking the orthogonal complement of the subspace. The well-known invariant of  $Z_2$ -spaces is homological index, introduced and studied in [9, 13, 3], see also the book [10] for a simplified introduction to the index and its many applications to combinatorics and geometry.

The following theorem gives an estimate for the index of the Grassmannian.

**Theorem 1.** *If  $n = 2^l(2m + 1)$ , then*

$$2^{l+1} - 1 \leq \text{ind } G_{2n}^n \leq 2n - 1,$$

*for  $n = 2m + 1$  the index equals 1, for  $n = 2(2m + 1)$  the index equals 3.*

The lower and the upper bounds coincide for  $n = 2^l$ , odd  $n$ ,  $n = 2(2m + 1)$ . In other cases there is still some gap between them. This result easily produces some geometric consequences. Here is one example (it also uses Lemma 1 below).

**Corollary 2.** *Let  $n = 2^l(2m + 1)$ ,  $k = 2^{l+1} - 1$ . Consider some  $k$  continuous (in the Hausdorff metric)  $O(n)$ -invariant functions  $\alpha_1, \dots, \alpha_k$*

---

2000 *Mathematics Subject Classification.* 52A38, 55M35, 55R25, 57S25.

*Key words and phrases.* Grassmannian, involution, homology index.

This research is supported by the Dynasty Foundation, the President's of Russian Federation grant MK-113.2010.1, the Russian Foundation for Basic Research grants 10-01-00096 and 10-01-00139.

on (convex) compacts in  $\mathbb{R}^n$ . Then for any (convex) compact  $K \subseteq \mathbb{R}^{2n}$  there exist a pair of orthogonal  $n$ -dimensional subspaces  $L$  and  $M$ , such that for their respective orthogonal projections  $\pi_L$  and  $\pi_M$  we have

$$\forall i = 1, \dots, k \quad \alpha_i(\pi_L(K)) = \alpha_i(\pi_M(K)).$$

In this corollary  $\alpha_i$  can be the Steiner measures (volume, the boundary measure, the mean width, etc.) for example. The same statement holds if we consider a point  $x \in K$  and sections of  $K$  by mutually orthogonal affine  $n$ -subspaces  $L$  and  $M$  through  $x$ , instead of projections to  $L$  and  $M$ .

The author thanks O.R. Musin for drawing attention to the problem of calculating the  $Z_2$ -index of  $G_{2n}^n$  and for the discussion.

## 2. PRELIMINARY OBSERVATIONS

Let us state some topological definitions on spaces with group action, see [6] for more detailed discussion.

**Definition 1.** Let  $G$  be a compact Lie group or a finite group. A space  $X$  with continuous action of  $G$  is called a  $G$ -space. A continuous map of  $G$ -spaces, commuting with the action of  $G$  is called a  $G$ -map or an *equivariant map*. A  $G$ -space is called *free* if the action of  $G$  is free.

There exists the universal free  $G$ -space  $EG$  such that any other  $G$ -space maps uniquely (up to  $G$ -homotopy) to  $EG$ . The space  $EG$  is homotopy trivial, the quotient space is denoted  $BG = EG/G$ . For any  $G$ -space  $X$  and an Abelian group  $A$  the equivariant cohomology  $H_G^*(X, A) = H^*(X \times_G EG, A)$  is defined, and for free  $G$ -spaces the equality  $H_G^*(X, A) = H^*(X/G, A)$  holds.

Consider the case  $G = Z_2$ . Note that

$$H_G^*(\text{pt}, Z_2) = H^*(\mathbb{R}P^\infty, Z_2) = Z_2[c] = \Lambda,$$

where the dimension of the generator is  $\dim c = 1$ . Since any  $G$ -space  $X$  can be mapped to the point  $\pi_X : X \rightarrow \text{pt}$ , we have a natural map  $\pi_X^* : \Lambda \rightarrow H_G^*(X, Z_2)$ , the image  $c$  under this map will be denoted by  $c$ , if it does not make a confusion. The generator element of  $Z_2$  will be denoted by  $\sigma$ .

**Definition 2.** The *cohomology index* of a  $Z_2$ -space  $X$  is the maximal  $n$  such that the power  $c^n \neq 0$  in  $H_G^*(X, Z_2)$ . If there is no maximum, we consider the index equal to  $\infty$ . Denote the index of  $X$  by  $\text{ind } X$ .

Let us state the following well-known lemma.

**Lemma 1** (The generalized Borsuk-Ulam theorem for odd maps). *If there exists an equivariant map  $f : X \rightarrow Y$ , then  $\text{ind } X \leq \text{ind } Y$ .*

Now we are ready to prove the upper bound in Theorem 1.

**Lemma 2.**

$$\text{ind } G_{2n}^n \leq 2n - 1.$$

*Proof.* Let us parameterize  $G_{2n}^n$  by the orthogonal projection matrices  $P$ . These matrices are characterized by the equations

$$P^t = P, \quad P^2 = P, \quad \text{tr } P = n.$$

The action of  $Z_2$  is given by ( $E$  is the unit matrix)

$$\sigma(P) = E - P.$$

Now consider the map  $f : G_{2n}^n \rightarrow \mathbb{R}^{2n}$ , defined by the coordinates

$$f_1(P) = P_{11} - 1/2, \quad f_i(P) = P_{1i}, \quad (i = 2, \dots, 2n).$$

This map is  $Z_2$ -equivariant, if the action on  $\mathbb{R}^{2n}$  is antipodal, i.e.  $\sigma : x \mapsto -x$ . Note also that  $f(P)$  is never zero, otherwise  $P$  would have an eigenvalue  $1/2$ , which is not true. Hence  $f$  composed with the projection  $\mathbb{R}^{2n} \setminus \{0\} \rightarrow S^{2n-1}$  gives the equivariant map

$$\tilde{f} : G_{2n}^n \rightarrow S^{2n-1},$$

and the result follows by Lemma 1.  $\square$

**Lemma 3.** Suppose  $n = ds$  for some positive integers  $d, s$ . Then

$$\text{ind } G_{2n}^n \geq \text{ind } G_{2d}^d.$$

*Proof.* Let us decompose

$$\mathbb{R}^{2n} = \mathbb{R}^{2d} \oplus \dots \oplus \mathbb{R}^{2d}$$

into  $s$  summands. Consider a  $d$ -subspace  $L \in G_{2d}^d$ , and define with the above decomposition

$$f(L) = L \oplus \dots \oplus L \subset \mathbb{R}^{2n}.$$

The map  $f : G_{2d}^d \rightarrow G_{2n}^n$  is evidently equivariant and by Lemma 1 we obtain the inequality.  $\square$

In order to prove Theorem 1 it remains to prove the following lemmas.

**Lemma 4.** If  $n$  is odd, then  $\text{ind } G_{2n}^n = 1$ , if  $n = 2 \pmod{4}$ , then  $\text{ind } G_{2n}^n = 3$ .

**Lemma 5.** If  $n = 2^l$ , then  $\text{ind } G_{2n}^n = 2n - 1$ .

### 3. EXTERNAL STEENROD SQUARES

In order to prove Lemma 5, we have to describe the cohomology of the subgroup  $G \subset O(2n)$ , generated by the subgroup  $O(n) \times O(n)$  (from some decomposition  $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ ), and  $Z_2$  that interchanges the summands  $\mathbb{R}^n$ . This group is the wreath product  $O(n) \wr Z_2 = (O(n) \times O(n)) \rtimes Z_2$ .

In order to describe the cohomology of a wreath product, we have to use the construction of external Steenrod squares. We mostly follow [2, Ch. V], where the Steenrod squares were defined in the unoriented cobordism. The cobordism was defined using mock bundles, if we allow the mock bundles to have codimension 2 singularities, we obtain ordinary cohomology modulo 2. In the sequel we consider the cohomology modulo 2 and omit the coefficients in notation. This construction is known and was used in [7] to describe the modulo 2 cohomology of the symmetric group and configuration spaces. Still, for reader's convenience we give a short and self-contained explanation here.

The construction of the external Steenrod squares on a polyhedron  $K$  starts with the fiber bundle (for some integer  $n > 0$ )

$$\sigma_K : (K \times K \times S^n)/Z_2 \rightarrow S^n/Z_2 = \mathbb{R}P^n.$$

The group  $Z_2$  acts by permuting  $K \times K$ , and antipodally on  $S^n$ . Consider a cohomology class  $\xi \in H^*(K)$ , represented by a mock bundle  $\xi : E(\xi) \rightarrow K$ . Then the mock bundle

$$(\xi \times \xi \times S^n)/Z_2 \rightarrow (K \times K \times S^n)/Z_2$$

is the external Steenrod square  $\text{Sq}_e \xi$ . The operation  $\text{Sq}_e$  is evidently multiplicative, in [2, Ch. V, Proposition 3.3] it is claimed that  $\text{Sq}_e$  is also additive. We are going to show that it is not true, first we need a definition.

**Definition 3.** The difference  $\text{Sq}_e(\xi + \eta) - \text{Sq}_e \xi - \text{Sq}_e \eta$  is represented by the mock bundle

$$\xi \odot \eta = (\xi \times \eta \times S^n + \eta \times \xi \times S^n)/Z_2,$$

where  $Z_2$  exchanges the components  $\xi \times \eta$  and  $\eta \times \xi$ .

Since the fiber of  $\sigma_K$  is  $K \times K$ , the restriction of  $\xi \odot \eta$  to the fiber is  $\xi \times \eta + \eta \times \xi$ , which is nonzero if  $\eta \neq \xi$  as cohomology classes. Thus the operation  $\odot$  is not trivial.

We need a lemma about the  $\odot$ -multiplication.

**Lemma 6.** Denote  $c$  the hyperplane class in  $H^1(\mathbb{R}P^n)$ . Then for any  $\xi, \eta \in H^*(K)$  the product

$$(\xi \odot \eta) \smile \sigma_K^*(c) = 0$$

in  $H^*((K \times K \times S^n)/Z_2)$ .

*Proof.* Consider the mock bundle

$$\alpha = \xi \times \eta \times S^{n-1} + \eta \times \xi \times S^{n-1},$$

which has the natural  $Z_2$ -action, it represents  $(\xi \odot \eta) \smile \sigma_K^*(c)$  after taking the quotient by the  $Z_2$ -action.

Now divide  $S^n$  into the upper and the lower half-spheres  $H^+$  and  $H^-$ . Consider the mock bundle (with boundary)

$$\beta = \xi \times \eta \times H^+ + \eta \times \xi \times H^-$$

over  $K \times K \times S^n$ . The action of  $Z_2$  on  $\beta$  is defined by permuting the summands and the antipodal identification of  $H^+$  and  $H^-$ . Now it is clear that  $\alpha$  is the boundary of  $\beta$ , and  $\alpha/Z_2$  is the boundary of  $\beta/Z_2$ . Hence it is zero in the cohomology, and the similar statement is true for the unoriented bordism.  $\square$

We have to introduce another operation.

**Definition 4.** Let  $\xi : E(\xi) \rightarrow K$ ,  $\eta : E(\eta) \rightarrow K$  be two mock bundles. Let  $p_+, p_-$  be the north and the south poles of  $S^n$ . Denote the mock bundle over  $(K \times K \times S^n)/Z_2$

$$\iota(\xi \times \eta) = (\xi \times \eta \times \{p_+\} + \eta \times \xi \times \{p_-\})/Z_2.$$

It is obvious from the definition that we have relation

$$\iota(\xi \times \eta) \smile \sigma_K^*(c) = 0,$$

it is also obvious that

$$\iota(\xi \times \xi) = \text{Sq}_e \xi \smile \sigma_K^*(c)^n.$$

Let us describe the  $\smile$ -multiplication of the Steenrod squares,  $\odot$ , and  $\iota(\dots)$  classes. The following formulas are obvious from the definition:

$$\begin{aligned} (\xi \odot \eta) \smile (\zeta \odot \chi) &= (\xi \smile \zeta) \odot (\eta \smile \chi) + (\xi \smile \chi) \odot (\eta \smile \zeta), \\ (\xi \odot \eta) \smile (\text{Sq}_e \zeta) &= (\xi \smile \zeta) \odot (\eta \smile \zeta), \\ (\xi \odot \eta) \smile \iota(\zeta \odot \chi) &= \iota((\xi \smile \zeta) \times (\eta \smile \chi)) + \iota((\xi \smile \chi) \times (\eta \smile \zeta)), \\ \text{Sq}_e \xi \smile \text{Sq}_e \eta &= \text{Sq}_e(\xi \smile \eta), \\ \text{Sq}_e \xi \smile \iota(\eta \times \zeta) &= \iota((\xi \smile \eta) \times (\xi \smile \zeta)), \\ \iota(\xi \times \eta) \smile \iota(\zeta \times \chi) &= 0. \end{aligned}$$

Now we can describe the structure of the cohomology  $H^*((K \times K \times S^n)/Z_2)$ .

**Definition 5.** Consider a  $Z_2$ -algebra  $A$  with linear basis  $v_1, \dots, v_n$ . Denote  $A \odot A$  the subalgebra of  $A \otimes A$ , invariant w.r.t.  $Z_2$ -action by permutation. The linear base of  $A$  is

$$\{v_i \otimes v_i\}_{i=1}^n, \{v_i \otimes v_j + v_j \otimes v_i\}_{i < j}.$$

**Definition 6.** Consider a  $Z_2$ -algebra  $A$  with linear basis  $v_1, \dots, v_n$ . Denote  $\iota(A \otimes A)$  the quotient vector space  $A \otimes A / (v_i \otimes v_j + v_j \otimes v_i)$ . As  $Z_2$ -algebra it has zero multiplication.

**Lemma 7.** *The maps  $\text{Sq}_e$ ,  $\odot$ , map the algebra  $H^*(K) \odot H^*(K)$  to  $H^*((K \times K \times S^n)/Z_2)$ . The map  $\iota$  maps  $\iota(H^*(K) \otimes H^*(K))$  to  $H^*((K \times K \times S^n)/Z_2)$ . The images of these maps generate the cohomology  $H^*((K \times K \times S^n)/Z_2)$ .*

*The latter cohomology can be described as the quotient of  $H^*(K) \odot H^*(K) \otimes Z_2[c] \oplus \iota(H^*(K) \otimes H^*(K))$  by the relations*

$$c^{n+1} = 0, (\xi \odot \eta) \otimes c = 0, \text{Sq}_e \xi \otimes c^n = \iota(\xi \otimes \xi).$$

*the  $c$  is the preimage of the hyperplane class in  $H^1(\mathbb{R}P^n)$ .*

Note the important particular case: if  $n \rightarrow \infty$ , we image of  $\iota(\dots)$  disappears, and we also can take the quotient of  $H^*(K) \odot H^*(K)$  by the linear span of all  $\xi \odot \eta$  for  $\xi, \eta \in H^*(K)$ . Hence, the cohomology  $H^*((K \times K \times S^\infty)/Z_2)$  has a quotient isomorphic to  $\text{Sq}_e(H^*(K)) \otimes Z_2[c]$ . Here  $\text{Sq}_e(H^*(K))$  is the same algebra as  $H^*(K)$ , but with twice larger degrees.

*Proof.* The Leray-Serre spectral sequence for  $\sigma_K$  starts with

$$E_2^{p,q} = H^p(\mathbb{R}P^n, \mathcal{H}^q(K \times K)).$$

Let us describe the sheaf  $\mathcal{H}^*(K \times K)$ . If  $v_1, \dots, v_n$  is the linear basis of  $H^*(K)$ , then an element  $v_i \otimes v_i$  gives a subsheaf, isomorphic to the constant sheaf  $Z_2$ . The two elements  $v_i \otimes v_j$  and  $v_j \otimes v_i$  generate a non-constant sheaf  $\mathcal{A} = Z_2 \oplus Z_2$  with permutation action of  $\pi_1(\mathbb{R}P^n)$ . The cohomology  $H^*(\mathbb{R}P^n, \mathcal{A}) = H^*(S^n, Z_2)$ , since  $\mathcal{A}$  is the direct image of  $Z_2$  under the natural projection  $\pi : S^n \rightarrow \mathbb{R}P^n$ . Thus we know the structure of  $E_2^{*,*}$ .

The first column of  $E_2$  is the  $Z_2$ -invariant elements of  $H^*(K \times K)$ , and all these elements are the restrictions of either  $\text{Sq}_e \xi$  or  $\xi \odot \eta$  to the fiber. Hence all the differentials of the spectral sequence are zero on the first column. The columns between the first and the last ( $n$ -th) are generated by multiplication with  $c$ , and the differentials are zero on them too. The last column is isomorphic to  $\iota(H^*(K) \otimes H^*(K))$ , the differentials are zero on it from the dimension considerations.

Hence in this spectral sequence  $E_2 = E_\infty$ . Denote  $v_1, \dots, v_n$  the linear base of  $H^*(K)$ . The first column of  $E_2$  has the linear base

$$\{v_i \times v_i\}_{i=1}^n, \{v_i \times v_j + v_j \times v_i\}_{i < j},$$

the columns No.  $j = 1, 2, \dots, n-1$  have the linear base

$$\{(v_i \times v_i)c^j\}_{i=1}^n,$$

and the last column has the linear base

$$\{\iota(v_i \times v_j)\}_{i,j=1}^n.$$

From the definition of  $\text{Sq}_e$ ,  $\odot$ , and  $\iota(\dots)$  the final cohomology  $H^*((K \times K \times S^n)/Z_2)$  is described the same way with  $v_i \times v_i$  replaced by  $\text{Sq}_e v_i$ , and  $v_i \times v_j + v_j \times v_i$  replaced by  $v_i \odot v_j$ .  $\square$

Now consider a vector bundle  $\nu : E(\nu) \rightarrow K$  and define

$$\text{Sq}_e \nu : (E(\nu) \times E(\nu) \times S^n)/Z_2 \rightarrow (K \times K \times S^n)/Z_2.$$

The Stiefel-Whitney classes of  $\text{Sq}_e \nu$  are described by the following lemma.

**Lemma 8.** *Let  $\dim \nu = k$ , and let the Stiefel-Whitney class of  $\nu$  be*

$$w(\nu) = w_0 + w_1 + \dots + w_k.$$

*Then*

$$w(\text{Sq}_e \nu) = \sum_{0 \leq i < j \leq k} w_i \odot w_j + \sum_{i=0}^k (1+c)^{k-i} \text{Sq}_e w_i,$$

*where  $c$  is the image of the hyperplane class in  $H^1(\mathbb{R}P^n)$ .*

*Proof.* Consider the case of one-dimensional  $\nu$  first. Taking  $n$  large enough we do not have to consider the image of  $\iota(\dots)$ , then we can return to lesser  $n$  by the natural inclusion

$$(K \times K \times S^n)/Z_2 \rightarrow (K \times K \times S^{n+m})/Z_2.$$

The restriction of  $\text{Sq}_e \nu$  to the fiber  $K \times K$  has the Stiefel-Whitney class

$$w(\nu \times \nu) = 1 + w_1(\nu) \times 1 + 1 \times w_1(\nu) + w_1(\nu) \times w_1(\nu).$$

Hence  $w(\text{Sq}_e \nu)$  is either  $1 + w_1(\nu) \odot 1 + \text{Sq}_e w_1(\nu)$ , or  $1 + w_1(\nu) \odot 1 + c + \text{Sq}_e w_1(\nu)$ . Any point  $x \in K$  gives a natural section

$$s : S^n/Z_2 \rightarrow (\{x\} \times \{x\} \times S^n)/Z_2$$

of the bundle  $\sigma_K$ , and the bundle  $s^*(\text{Sq}_e \nu)$  over  $\mathbb{R}P^n$  is isomorphic to  $\gamma \oplus \varepsilon$ , where  $\gamma$  is the canonical bundle of the projective space,  $\varepsilon$  is the trivial bundle. Hence we should have

$$w(\text{Sq}_e \nu) = 1 + w_1(\nu) \odot 1 + c + \text{Sq}_e w_1(\nu).$$

The general formula for  $k > 1$  follows from the splitting principle, suppose  $\nu = \tau_1 \oplus \dots \oplus \tau_k$ , then

$$w(\text{Sq}_e \nu) = \prod_{i=1}^k (1 + w_1(\tau_i) \odot 1 + c + \text{Sq}_e w_1(\tau_i)),$$

and the result follows by removing parentheses. □

#### 4. THE PROOF OF LEMMAS 4 AND 5

In order to calculate the index of  $G_{2n}^n$ , we describe the cohomology of  $G_{2n}^n/Z_2$ . Consider the subgroup  $G = O(n) \wr Z_2$  of  $O(2n)$ , that is generated by two copies of  $O(n)$  for some decomposition  $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ , and by the operator  $\sigma$  that interchanges the summands of the decomposition. It is clear that  $G_{2n}^n/Z_2 = O(2n)/G$ .

The cohomology of  $BO(n)$  is the polynomial algebra in Stiefel-Whitney classes

$$H^*(BO(n)) = Z_2[w_1, \dots, w_n].$$

The group cohomology  $H^*(BG)$  (by Lemma 7) is generated by the external Steenrod squares  $Sq_e w_1, \dots, Sq_e w_n$ , the generator  $c \in H^1(BZ_2)$ , and some combinations  $x \odot y$  for  $x, y \in H^*(BO(n))$ , the relations are  $(x \odot y)c = 0$ .

Let us find the kernel of the natural map  $\pi^* : H^*(BG) \rightarrow H^*(O(2n)/G)$ . The cohomology  $H^*(O(2n)/G)$  can be calculated by considering the Leray-Serre spectral sequence with the term  $E_2^{p,q} = H^p(BG, \mathcal{H}^q(O(2n)))$ , see [11, Section 11.4]. The kernel of  $\pi^*$  is given by the images of the differentials  $d_r$  of this spectral sequence in its bottom row.

Note that the action of  $G$  on  $O(2n)$  is induced by the inclusion  $G \in O(2n)$ , and the cohomology of  $O(2n)$  is acted on by  $G$  through its factor group  $G/G^+$  of order 2. Here  $G^+$  denotes the elements of  $G$  with positive determinant. Hence we can replace  $G$  by  $G^+$  and simultaneously pass from the sheaf  $\mathcal{H}^q(O(2n))$  to the cohomology  $H^q(SO(2n))$  (see [1], Ch. III, Proposition 6.2), thus obtaining

$$E_2^{p,q} = H^p(BG, \mathcal{H}^q(O(2n))) = H^p(BG^+, H^q(SO(2n))).$$

In order to find the images of  $d_r$ 's, note that the fiber bundle

$$\begin{array}{ccc} SO(2n) & \longrightarrow & SO(2n) \times_G EG^+ \\ & & \downarrow \\ & & BG^+ \end{array}$$

is induced from the fiber bundle

$$\begin{array}{ccc} SO(2n) & \longrightarrow & ESO(2n) \\ & & \downarrow \\ & & BSO(2n) \end{array}$$

by the inclusion  $G^+ \rightarrow SO(2n)$ . In the spectral sequence of the latter fiber bundle all the primitive generators of  $H^*(SO(2n))$  are transgressive. They are mapped to the bottom row by the corresponding differentials  $d_r$ , their images being the Stiefel-Whitney classes of  $O(2n)$ . Thus, in the considered spectral sequence, the differentials  $d_r$  are generated by the transgressions that send the primitive generators of  $H^*(SO(2n))$  to the Stiefel-Whitney classes of the representation of  $G^+$  on  $\mathbb{R}^{2n}$ . Denote this representation  $W_{2n}$ .

Let us summarize as follows.

**Lemma 9.** *The kernel of the natural map  $\pi^* : H^*(BG) \rightarrow H^*(O(2n)/G)$  is generated by the homogeneous components of positive degree of the*



expression

$$\sum_{0 \leq i < j \leq n} w_i \odot w_j + \sum_{i=0}^n (1+c)^{n-i} \text{Sq}_e w_i.$$

*Proof.* In the bottom row of the spectral sequence passing from  $H^*(BG)$  to  $H^*(BG^+)$  “kills” the element  $w_1(W_{2n})$  and the ideal generated by it. The other differentials “kill” the other classes  $w_r(W_{2n})$  by the above considerations.

It remains to calculate the Stiefel-Whitney classes of  $W_{2n}$ . Remind that by the Stiefel-Whitney classes of a representation we mean the Stiefel-Whitney classes of the vector bundle  $\eta : (W_{2n} \times EG)/G \rightarrow BG$ . Denote  $V_n$  the natural representation of  $O(n)$ , and consider its corresponding bundle  $\xi : (V_n \times EO(n))/O(n) \rightarrow BO(n)$ . It can be checked by definition that  $\eta = \text{Sq}_e \xi$  and the claim follows by applying Lemma 8.  $\square$

Now the proof of Lemma 4 is finished as follows: we have to find the nilpotency degree of  $c$  in  $H^*(BG)/\ker \pi^*$ . If  $n$  is odd, then the one-dimensional generator of  $\ker \pi^*$  is

$$c + w_1 \odot 1,$$

hence  $c \neq 0$ ,  $c^2 = 0$  by Lemma 6, and  $\text{ind } G_{2n}^n = 1$  in this case.

If  $n \equiv 2 \pmod{4}$ , then we have the relations in dimensions 2 and 3

$$\begin{aligned} c^2 + \text{Sq}_e w_1 + 1 \odot w_2 &= 0 \\ c \text{Sq}_e w_1 + 1 \odot w_3 + w_1 \odot w_2 &= 0. \end{aligned}$$

Substituting  $\text{Sq}_e w_1 = c^2 + 1 \odot w_2$  from the first relation to the second we obtain

$$c^3 = 1 \odot w_3 + w_1 \odot w_2,$$

hence  $c^4 = 0$  by Lemma 6, and  $\text{ind } G_{2n}^n = 3$  in this case.

Now let us turn to Lemma 5. Let  $n = 2^l$ , and let us add the additional relations of the form  $w_i = 0$  for all  $i$  except  $i = 2^l - 2^k$  ( $k = 0, \dots, l$ ) and  $i = 2^l$ . In this case the remaining relations in  $\ker \pi^*$  are

$$\begin{aligned} c^{2^l} &= \text{Sq}_e w_{2^l-2^{l-1}} + 1 \odot w_{2^l} \\ c^{2^{l-1}} \text{Sq}_e w_{2^l-2^{l-1}} &= \text{Sq}_e w_{2^l-2^{l-2}} + w_{2^l-2^{l-1}} \odot w_{2^l} \\ &\dots \\ c^2 \text{Sq}_e w_{2^l-2} &= \text{Sq}_e w_{2^l-1} + w_{2^l-2} \odot w_{2^l} \\ c \text{Sq}_e w_{2^l-1} &= w_{2^l-1} \odot w_{2^l} \\ \text{Sq}_e w_{2^l} &= 0, \end{aligned}$$

along with the relations of the form

$$w_{2^l-2^k} \odot w_{2^l-2^m} = 0, \quad 0 \leq k < m \leq l.$$

Thus we obtain  $c^{2^{l+1}-1} = c^{2n-1} = w_{2^l} \odot w_{2^l-1} \neq 0$ . Also, we must have  $c^{2n} = 0$  by the upper bound  $\text{ind } G_{2n}^n \leq 2n - 1$ , without any additional relations. Therefore,  $\text{ind } G_{2n}^n = 2n - 1$  in this case.

## REFERENCES

- [1] K. Brown. Cohomology of groups. Graduate Texts in Mathematics, 87, New York: Springer-Verlag, 1982.
- [2] S. Buoncristiano, C.P. Rourke, B.J. Sanderson. A geometric approach to homology theory. Cambridge University Press, 1976.
- [3] P.E. Conner, E.E. Floyd. Fixed point free involutions and equivariant maps. // Bull. Amer. Math. Soc., 66(6), 1960, 416–441.
- [4] V.L. Dol’nikov. Transversals of families of sets in  $\mathbb{R}^n$  and a connection between the Helly and Borsuk theorems (In Russian). // Sb., Math. 79(1), 1994, 93–107; translation from Mat. Sb., 184(5), 1993, 111–132.
- [5] H.L. Hiller. On the cohomology of real grassmanians. // Trans. Amer. Math. Soc., 257(2), 1980, 521–533.
- [6] Wu Yi Hsiang. Cohomology theory of topological transformation groups. Berlin-Heidelberg-New-York, Springer Verlag, 1975.
- [7] Nguyễn H.V. Hung. The mod 2 equivariant cohomology algebras of configuration spaces. // Pacific Jour. Math., 143(2), 1990, 251–286.
- [8] R.N. Karasev. Theorems of Borsuk-Ulam type for flats and common transversals (In Russian). // Math. Sbornik, 200(10), 2009, 39–58; translated in arXiv:0905.2747.
- [9] M.A. Krasnosel’skii. On the estimation of the number of critical points of functionals (In Russian). // Uspehi Mat. Nauk, 7(2), 1952, 157–164.
- [10] J. Matoušek. Using the Borsuk-Ulam theorem. // Berlin-Heidelberg, Springer Verlag, 2003.
- [11] J. McCleary. A user’s guide to spectral sequences. Cambridge University Press, 2001.
- [12] L. Montejano, R.N. Karasev. Topological transversals to a family of convex sets. // arXiv:1006.0104, 2010.
- [13] A.S. Schwarz. Some estimates of the genus of a topological space in the sense of Krasnosel’skii. (In Russian) // Uspehi Mat. Nauk, 12:4(76), 1957, 209–214.
- [14] N.E. Steenrod, D.B. Epstein. Cohomology operations. Princeton University Press, 1962.
- [15] R. Živaljević. Topological methods. // Handbook of Discrete and Computational Geometry, ed. by J.E. Goodman, J. O’Rourke, CRC, Boca Raton, 2004.

*E-mail address:* `r.n.karasev@mail.ru`

ROMAN KARASEV, DEPT. OF MATHEMATICS, MOSCOW INSTITUTE OF PHYSICS AND TECHNOLOGY, INSTITUTSKIY PER. 9, DOLGOPRUDNY, RUSSIA 141700